

EXERCISES for Chapter 2: Matrices

1. Write the following systems of equations as augmented matrices and hence find the solutions

$$(a) \begin{cases} 2x + y = 4 \\ 5x - 2y = 1 \end{cases} \quad (b) \begin{cases} x + 5y + z = 0 \\ 2y + 3z = 4 \\ 2x - 2y - z = 6 \end{cases}$$

$$(c) \begin{cases} 14 - x + 5y = 0 \\ 7z + 2x - 15 = 0 \\ x - y + 3z = 9 \end{cases} \quad (d) \begin{cases} x - 2y = 4 \\ 2z + w = 5 \\ w + y = 2 \\ 3x - 5z = 1 \end{cases} \quad (e) \begin{cases} 3x - z = 1 \\ 2y + z - 3u + 2v = 2 \\ z - u + 4v = 3 \\ 4x + 2y + u - v = 0 \\ x + 4y = 2 \end{cases}$$

2. Solve the following sets of equations. Comment on each case (e.g. the number of equations and unknowns, whether there is inconsistency or linear dependence, whether the solutions are unique, many, trivial, etc.).

$$(a) \begin{cases} 2x + y - z = 2 \\ 4x + y - 2z = 3 \end{cases} \quad (c) \begin{cases} x - y + 2z = 5 \\ 2x + 3y - z = 4 \\ 2x - 2y + 4z = 6 \end{cases} \quad (e) \begin{cases} 3x + y + 3z + 6w = 0 \\ 4x - 7y - 3z + 5w = 0 \\ x + 3y + 4z - 3w = 0 \\ 3x + 2z + 7w = 0 \end{cases}$$

$$(b) \begin{cases} 2x + 3y = 1 \\ x + 2y = 2 \\ x + 3y = 5 \end{cases} \quad (d) \begin{cases} x - 2y + 3z = 0 \\ x + 4y - 6z = 0 \\ 2x + 2y - 3z = 0 \end{cases} \quad (f) \begin{cases} x - 3y - 2z + 3v = 1 \\ z + 3u + 5v = -1 \\ x - 3y + 3z + u + 2v = 6 \\ z + u + v = 1 \end{cases}$$

3. For what value of a does the system of equations shown have solutions? Find those solutions.
- $$\begin{aligned} x + y + z &= 3 \\ x - y + 3z &= 5 \\ 3x - y + 7z &= a \end{aligned}$$

4. $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 5 & 3 \\ 4 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$.

State whether the following exist. Find those which do exist.

- (a) $A + B$ (b) $B + D$ (c) $3A^T - C$ (d) AB (e) BA
 (f) $A^2 = AA$ (g) AA^T (h) B^3 (i) $BACD$

5. For matrices A and C of Question 4, demonstrate that $(AC)^T = C^T A^T$.

6. Write down an example of (a) a row matrix, (b) a 2×5 matrix, (c) a diagonal matrix of order 4, (d) a symmetric matrix of order 3. [Do NOT use examples from the lectures!]
7. Find (by guesswork or otherwise) an example of two matrices which commute and have a non-zero product. (I.e. find two matrices A and B for which $AB = BA \neq 0$). Do not use the identity matrix.
8. For the matrices in Question 4, state whether the following exist. Find those which do exist. (a) $|A|$ (b) A^{-1} (c) $|B|$ (d) B^{-1} (e) $|D|$ (f) D^{-1} .
9. Evaluate the following determinants

$$(a) \begin{vmatrix} 1 & 2 & 0 \\ 3 & -2 & 4 \\ 2 & 0 & -1 \end{vmatrix}, \quad (b) \begin{vmatrix} 3 & -2 & 4 \\ 0 & 1 & -1 \\ 5 & 7 & 2 \end{vmatrix}, \quad (c) \begin{vmatrix} 2 & 1 & k \\ -1 & 5 & 2 \\ 2 & -1 & 4 \end{vmatrix}.$$

10. For $A = \begin{bmatrix} 3 & -6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & \frac{1}{2} \\ 2 & \frac{1}{2} \end{bmatrix}$, (a) verify that $|A||B| = |AB|$
 (b) verify that $(AB)^{-1} = B^{-1}A^{-1}$

11. (a) Find the inverse of matrix A and (b) use it to solve the given system of equations.

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \begin{aligned} 3x + 2y &= 8, \\ -2x + y - z &= 7, \\ x - y + z &= 5 \end{aligned}$$

12. Solve the system of equations shown by using the inverse of the coefficient matrix. (To check your answers, see lecture example 3!)
- $$\begin{aligned} 2x_1 + x_2 + x_3 &= -2 \\ -x_2 + 3x_3 &= -8 \\ x_1 + 4x_2 - x_3 &= 9 \end{aligned}$$

13. Find the inverses of the following matrices:

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

14. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ -12 & 0 & 1 \end{bmatrix}.$$

Application Questions

15. Solve the animal feed problem at the beginning of the lecture notes using an augmented matrix.
16. A lake is to be stocked with three species of fish: A, B and C. Three foods I, II and III are available in the lake. On average, each day each fish of species A requires 1 unit of food I, 4 units of food II and 3 units of food III. Fish B require 2, 1 and 5 units of foods I, II and III respectively. For fish C the figures are 4, 1 and 2 units. If a total of 250 units of food I, 155 units of food II and 315 units of food III are available daily, how many of each species of fish should be stocked?
17. The gross profits of a certain company are \$115,000. The manager of a business is to receive a bonus amounting to 20% of the profits which remain after deducting the business's income tax (but not the manager's bonus) from the gross profits. The income tax is 40% of the profits which remain after deducting the manager's bonus (but not the tax) from the gross profits. Find the bonus and the tax.
18. A dietician prescribes a special diet for a patient involving three types of food, A, B and C, according to the following rules:
 - (i) The total amount of food per day must be 900 grams.
 - (ii) The amount of food A should be one-third the amount of food B.
 - (iii) The sum of the amounts of A and C should equal twice the amount of B.
 - (a) Find the amount needed of each food group.
 - (b) If requirement (ii) is dropped, describe all the possible solutions.
19. A dealer sells brand X and brand Y washing machines. Below is a matrix showing how many machines of each type were sold in each of three months and another matrix showing the retail (sale) price and the dealer cost of each brand. Use matrix multiplication to find the total sales income and the total cost to the dealer for each month.

	Jan	Feb	Mar		X	Y
Brand X	18	10	12	Retail price	350	260
Brand Y	19	12	14	Dealer cost	240	190

20. A dietician prescribes a special diet involving three food groups: I (meat), II (fruit and vegetables) and III (bread and rice).
 - (a) According to this diet, for breakfast a person should eat 2 portions of food group I, 1 portion of group II and 2 portions of group III. For lunch the figures should be 3, 2 and 2 portions respectively. For supper, 4, 3 and 2 portions. Write this information in a matrix A, labelling the rows and columns.
 - (b) On average, the number of units of fat, carbohydrate and protein in a portion of food are as follows for groups I, II and II respectively:
 Fat: 5, 0, 0; Carbohydrate: 0, 8, 14. Protein are 7, 1, 2.
 Write this information in a matrix, B, labelling the rows and columns.
 - (c) 1 unit of fat provides 8 calories, a unit of carbohydrate provides 4 calories, and 1 unit of protein provides 5 calories. Write this information in a labelled matrix C.
 - (d)-(g) Find a matrix showing each of the following:
 - (d) the units of fat, carbohydrate and protein consumed at each meal.
 - (e) the number of calories in a portion of food of each group.
 - (f) the total calories consumed at each meal.
 - (g) total units of fat, carbohydrate and protein eaten each day.
 - (h) Find the total calorie consumption per day.

[Note: Depending on how you write your matrices A, B and C, you may need to *transpose* one or more matrix before multiplying.]
21. For the following Leslie matrices, find a population of total size 3000 for which the proportion in each age group remains unchanged from one year to the next. State also the factor by which the population grows or declines each year.

$$(a) \begin{bmatrix} 0.7 & 0.9 \\ 1.6 & 0 \end{bmatrix}, \quad (b) \begin{bmatrix} 0.8 & 0.4 \\ 0.5 & 0 \end{bmatrix}.$$
